

A Mehler–Heine formula for disk polynomials

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ABSTRACT

We prove a formula of Mehler–Heine type for a family of orthogonal polynomials in two variables on the unit disk. As an application we obtain a non-classical two dimensional central limit theorem for the iterates of a Markovian matrix which is closely linked with these polynomials.

1. INTRODUCTION AND SUMMARY OF THE RESULTS

For arbitrary real numbers α and β , Jacobi polynomials $R_n^{(\alpha, \beta)}$ normalized by $R_n^{(\alpha, \beta)}(1) = 1$ satisfy

$$(1.1) \quad \lim_{n \rightarrow \infty} R_n^{(\alpha, \beta)} \left(\cos \frac{\theta}{n} \right) = 2^\alpha \Gamma(\alpha + 1) \theta^{-\alpha} J_\alpha(\theta),$$

uniformly for θ in every bounded interval. (1.1) is the Mehler–Heine formula ([10], 8.1.1) and

$$(1.2) \quad J_\alpha(x) = \sum_{i=0}^{+\infty} \frac{(-1)^i (x/2)^{\alpha+2i}}{i! \Gamma(i + \alpha + 1)}$$

is the Bessel function of the first kind and index α . If α is a negative integer, $(\Gamma(i + \alpha + 1))^{-1}$ must be replaced by 0 whenever $i + \alpha + 1 \leq 0$.

In this paper we prove a similar formula for the so-called disk polynomials ([2], [6], [9]). For every $\alpha \geq 0$, they form a family $(R_{m,n}^\alpha(z, \bar{z}))_{(m,n) \in \mathbb{N}^2}$ of polynomials in two variables normalized by $R_{m,n}^\alpha(1, 1) = 1$ and are defined by orthogonalization of the sequence $1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2, \dots$ on the unit disk $D = \{z \in \mathbb{C} \mid |z| \leq 1\}$ with respect to the measure

$$(1.3) \quad \lambda_\alpha(dx dy) = \frac{\alpha+1}{\pi} (1-x^2-y^2)^\alpha dx dy \quad (z=x+iy \in D).$$

The orthogonality relations are given by

$$(1.4) \quad \frac{\alpha+1}{\pi} \iint_D R_{m,n}^\alpha(z) \overline{R_{k,l}^\alpha(z)} (1-x^2-y^2)^\alpha dx dy = (\Pi_{m,n}^\alpha)^{-1} \delta_{m,k} \delta_{n,l},$$

with

$$(1.5) \quad \Pi_{m,n}^\alpha = \frac{(m+n+\alpha+1)\Gamma(m+\alpha+1)\Gamma(n+\alpha+1)}{m! n! \Gamma(\alpha+1)\Gamma(\alpha+2)},$$

and where we have written $R_{m,n}^\alpha(z) := R_{m,n}^\alpha(z, \bar{z})$ to simplify notations.

Introducing polar coordinates, $R_{m,n}^\alpha$ can be represented in terms of Jacobi polynomials by the formula

$$(1.6) \quad R_{m,n}^\alpha(\varrho e^{i\theta}) = e^{i(m-n)\theta} \varrho^{|m-n|} R_{m \wedge n}^{(\alpha, |m-n|)}(2\varrho^2 - 1),$$

where $m \wedge n = \min(m, n)$ ([2], [6]). A trivial but useful consequence of (1.6) is:

$$(1.7) \quad R_{m,n}^\alpha(\varrho e^{i\theta}) = e^{i(m-n)\theta} R_{m,n}^\alpha(\varrho).$$

The main result of this paper is the following theorem.

THEOREM 1. *Let $\alpha \geq 0$ and let $C > 0$ be arbitrary. Then there exist real numbers $N_c > 0$, $C_1 > 0$ and $C_2 > 0$ such that for every $(m, n) \in \mathbb{N}^2$ satisfying $mn \geq N_c$ and for every real number $\psi \in]0, C/2 \sqrt{mn}]$, we have*

$$(1.8) \quad \left(\frac{\sin \psi}{\psi} \right)^{\alpha+1/2} (\cos \psi)^{1/2} R_{m,n}^\alpha(\cos \psi) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(\beta_{mn}\psi)}{(\beta_{mn}\psi)^\alpha} + r(m, n; \psi)$$

where

$$(1.9) \quad \beta_{mn} = ((2m+\alpha+1)(2n+\alpha+1))^{1/2}$$

and

$$(1.10) \quad |r(m, n; \psi)| \leq C_1 \psi^2 + C_2 (m-n)^2 \psi^4 \quad \text{for } \alpha > 0.$$

$$(1.10)' \quad |r(m, n; \psi)| \leq (C_1 \psi^2 + C_2 (m-n)^2 \psi^4) (|\text{Log}(\beta_{mn}\psi)| + 1) \quad \text{for } \alpha = 0.$$

COROLLARY (Mehler-Heine formula). Let $\alpha \geq 0$. Then

$$(1.11) \quad \lim_{\substack{n \rightarrow \infty \\ m \rightarrow \infty}} R_{m,n}^\alpha \left(\cos \frac{x}{2\sqrt{mn}} \right) = 2^\alpha \Gamma(\alpha+1) x^{-\alpha} J_\alpha(x),$$

and this limit holds uniformly in every bounded interval.

The proof of Theorem 1 is carried out in section 2. Let's now mention recurrence relations which can be easily deduced from (1.6) by using formulas 10.8 (33) and (36) of [4]:

$$(1.12) \quad z R_{m,n}^\alpha(z) = \frac{\alpha+m+1}{\alpha+m+n+1} R_{m+1,n}^\alpha(z) + \frac{n}{\alpha+m+n+1} R_{m,n-1}^\alpha(z)$$

for every $m \geq 0$, $n \geq 1$, and

$$(1.12)' \quad \bar{z} R_{m,n}^\alpha(z) = \frac{\alpha + n + 1}{\alpha + m + n + 1} R_{m,n+1}^\alpha(z) + \frac{m}{\alpha + m + n + 1} R_{m-1,n}^\alpha(z)$$

for every $m \geq 1$, $n \geq 0$.

We can then define a transition matrix $P = (p((m, n), (i, j)))$ on \mathbb{N}^2 by

$$(1.13) \quad p((m, n), (i, j)) = \begin{cases} \frac{1}{2} \frac{\alpha + m + 1}{\alpha + m + n + 1} & \text{if } (i, j) = (m + 1, n) \\ \frac{1}{2} \frac{m}{\alpha + m + n + 1} & \text{if } (i, j) = (m - 1, n) \text{ and } m \geq 1 \\ \frac{1}{2} \frac{\alpha + n + 1}{\alpha + m + n + 1} & \text{if } (i, j) = (m, n + 1) \\ \frac{1}{2} \frac{n}{\alpha + m + n + 1} & \text{if } (i, j) = (m, n - 1) \text{ and } n \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

By adding relations (1.12) and (1.12)' and then multiplying by $\frac{1}{2}$, we get a generating relation for P :

$$(1.14) \quad x R_{m,n}^\alpha(z) = \sum_{i,j=0}^{+\infty} p((m, n), (i, j)) R_{i,j}^\alpha(z) \quad (x = \operatorname{Re} z).$$

Next multiply (1.14) by x and develop $x R_{i,j}^\alpha(z)$ in the same manner. We have

$$\begin{aligned} x^2 R_{m,n}^\alpha(z) &= \sum_{i,j=0}^{+\infty} p((m, n), (i, j)) \sum_{k,l=0}^{+\infty} p((i, j), (k, l)) R_{k,l}^\alpha(z) \\ &= \sum_{k,l=0}^{+\infty} \left(\sum_{i,j=0}^{+\infty} p((m, n), (i, j)) p((i, j), (k, l)) \right) R_{k,l}^\alpha(z) \\ &= \sum_{k,l=0}^{+\infty} p^{(2)}((m, n), (k, l)) R_{k,l}^\alpha(z), \end{aligned}$$

where for every $N \in \mathbb{N}$, $p^{(N)}((m, n), (k, l))$ denotes the coefficient $(m, n), (k, l)$ of the matrix P^N . After $N - 1$ iterations of the procedure of multiplying by x we finally obtain

$$(1.15) \quad x^N R_{m,n}^\alpha(z) = \sum_{k,l=0}^{+\infty} p^{(N)}((m, n), (k, l)) R_{k,l}^\alpha(z),$$

and we can also recover the coefficient $p^{(N)}((m, n), (k, l))$ by multiplying equation (1.15) by $\overline{R_{k,l}^\alpha(z)}$ and integrating with respect to λ_α over D . The orthogonality relations imply

$$(1.16) \quad p^{(N)}((m, n), (k, l)) = \Pi_{k,l}^\alpha \iint_D x^N R_{m,n}^\alpha(z) \overline{R_{k,l}^\alpha(z)} \lambda_\alpha(dx dy).$$

For every subset A of $\mathbb{R}_+ \times \mathbb{R}_+$, we define

$$(1.17) \quad P^N((m, n), A) = \sum_{(k, l) \in A} p^{(N)}((m, n), (k, l)).$$

We are now ready to present a two dimensional central limit theorem which is the second main result of this paper:

THEOREM 2. *Let $A = [a, b] \times [c, d]$ (with $a > 0, c > 0$) a rectangular subset of $\mathbb{R}_+ \times \mathbb{R}_+$ and let (m_0, n_0) be an arbitrary point in \mathbb{N}^2 . Then*

$$\lim_{N \rightarrow +\infty} P^N((m_0, n_0), \sqrt{N}A) = \frac{2^{\alpha+1}}{\sqrt{2\pi} \Gamma(\alpha+1)} \iint_A (xy)^\alpha (x+y) e^{-(1/2)(x+y)^2} dx dy.$$

One dimensional central limit theorems associated to orthogonal polynomials in one variable have already been obtained in the case of Gegenbauer polynomials ([5]) and recently for some orthogonal families including Jacobi polynomials ([11]).

2. PROOF OF THE THEOREM 1

2.1. LEMMA. *For every $\alpha \geq 0$ and every $(m, n) \in \mathbb{N}^2$, we have*

$$\sup_{z \in D} |R_{m,n}^\alpha(z)| = 1.$$

PROOF. We follow an idea of R. Askey ([1] (2.2)). By a result of T. Koornwinder [7], disk polynomials satisfy a linearization formula

$$R_{m_1, n_1}^\alpha(z) R_{m_2, n_2}^\alpha(z) = \sum_{(m, n)} C^\alpha(m_1, n_1, m_2, n_2; m, n) \overline{R_{m, n}^\alpha(z)}$$

where the coefficients C^α are non-negative for every (m_1, n_1) , (m_2, n_2) and (m, n) . Then if $M(z) = \sup_{(m, n)} |R_{m, n}^\alpha(z)|$ and $M = \sup_{z \in D} M(z)$, we have $M^2(z) \leq M$ since $R_{m, n}^\alpha(1) = 1$. Thus $M^2 \leq M$, so $M = 1$ and the lemma is proved.

2.2. The function $u(\theta) = (\sin \theta)^{\alpha+1/2} (\cos \theta)^{\beta+1/2} R_n^{(\alpha, \beta)}(\cos 2\theta)$ satisfies the differential equation

$$\frac{d^2 u}{d\theta^2} + \left[\frac{\frac{1}{4} - \alpha^2}{\sin^2 \theta} + \frac{\frac{1}{4} - \beta^2}{\cos^2 \theta} + (2n + \alpha + \beta + 1)^2 \right] u(\theta) = 0$$

([10] 4.24.2). This shows by (1.6), that the function

$$(2.3) \quad v_{m, n}(\psi) = (\sin \psi)^{\alpha+1/2} (\cos \psi)^{1/2} R_{m, n}^\alpha(\cos \psi)$$

is a solution of the differential equation

$$(2.4) \quad \frac{d^2 v}{d\psi^2} + \left[\frac{\frac{1}{4} - \alpha^2}{\sin^2 \psi} + \frac{\frac{1}{4} - (m - n)^2}{\cos^2 \psi} + (n + m + \alpha + 1)^2 \right] v(\psi) = 0.$$

The main point is to rewrite (2.4) as follows

$$(2.5) \quad \frac{d^2 v}{d\psi^2} + \left[\frac{\frac{1}{4} - \alpha^2}{\psi^2} + \beta_{mn}^2 \right] v(\psi) = [f(\psi) + (m - n)^2 g(\psi)] v(\psi)$$

where

$$\begin{aligned}\beta_{mn}^2 &= (2m + \alpha + 1)(2n + \alpha + 1), \\ f(\psi) &= \left(\frac{1}{4} - \alpha^2\right) \left(\frac{1}{\psi^2} - \frac{1}{\sin^2 \psi}\right) - \frac{1}{4 \cos^2 \psi} \\ g(\psi) &= \frac{1}{\cos^2 \psi} - 1.\end{aligned}$$

The general solution of (2.5) is classical ([10] 1.8.12). We have

$$(2.6) \quad \begin{cases} v_{m,n}(\psi) = A_1 \psi^{1/2} J_\alpha(\beta_{mn} \psi) + A_2 \psi^{1/2} J_{-\alpha}(\beta_{mn} \psi) \\ \quad + \frac{\psi^{1/2}}{\beta_{mn}} \int_0^\psi t^{-1/2} \frac{J_\alpha(\beta_{mn} \psi) J_{-\alpha}(\beta_{mn} t) - J_{-\alpha}(\beta_{mn} \psi) J_\alpha(\beta_{mn} t)}{J'_\alpha(\beta_{mn} t) J_{-\alpha}(\beta_{mn} t) - J'_{-\alpha}(\beta_{mn} t) J_\alpha(\beta_{mn} t)} \\ \quad \cdot (f(t) + (m-n)^2 g(t)) v_{m,n}(t) dt, \end{cases}$$

where A_1 and A_2 are arbitrary constants, and $J_{-\alpha}(x)$ must be replaced by $Y_\alpha(x)$, the Bessel function of the second kind for $\alpha \in \mathbb{N}$ (in this way if $\alpha = 0$, J_{-0} denotes Y_0). Moreover we know ([10] 1.8.14) that

$$(2.7) \quad J'_\alpha(\beta_{mn} t) J_{-\alpha}(\beta_{mn} t) - J'_{-\alpha}(\beta_{mn} t) J_\alpha(\beta_{mn} t) = \frac{\gamma}{\beta_{mn} t},$$

where $\gamma = 2 \sin \alpha \pi / \pi$ for $\alpha \notin \mathbb{N}$ and $\gamma = -2/\pi$ for $\alpha \in \mathbb{N}$.

We are now going to estimate the last term on the right-hand side of (2.6) by a method of Liouville–Stekloff type. Let $C > 0$ a fixed number and let $M_c > 0$ be such that $C/M_c < \pi/2$. Let's write $(m, n; \psi) \in \mathcal{D}$ to indicate that $(m, n) \in \mathbb{N}^2$ satisfies $4mn \geq M_c^2$ and $\psi \in \mathbb{R}_+$ is such that $\beta_{mn} \psi \leq C$. In particular $\psi \in [0, C/M_c]$ if $(m, n; \psi) \in \mathcal{D}$ and on $[0, \psi]$, we have

$$(2.8) \quad \begin{cases} f(t) = \gamma_0(1 + O(1)) & (\gamma_0 = \lim_{t \rightarrow 0} f(t)) \\ g(t) = t^2(1 + O(1)) \\ v_{m,n}(t) = t^{\alpha+1/2}(1 + O(1)) \end{cases}$$

where $O(1)$ denotes a function of t bounded by a constant independent of $(m, n; t)$ (this is clear for f and g and this is a consequence of Lemma 2.1 for $v_{m,n}$). But we have also

$$(2.9) \quad J_\nu(\beta_{mn} t) = \lambda_\nu (\beta_{mn} t)^\nu (1 + O(1)) \quad (\nu = \pm \alpha)$$

$$(2.9)' \quad Y_0(\beta_{mn} t) = \frac{2}{\pi} \text{Log}(\beta_{mn} t) (1 + O(1)),$$

with λ_ν a constant and $O(1)$ a function of t (in fact of $\beta_{mn} t$) bounded independently of $(m, n; t) \in \mathcal{D}$.

Taking into account (2.7), (2.8) and (2.9) and integrating over ψ we obtain the following expression for the remainder term in (2.6):

$$(2.10) \quad \begin{cases} \left(\psi^{1/2} \frac{J_\alpha(\beta_{mn}\psi)}{(\beta_{mn})^\alpha} (\gamma_1 \psi^2 + (m-n)^2 \gamma_2 \psi^4) \right. \\ \left. - \psi^{1/2} \frac{J_{-\alpha}(\beta_{mn}\psi)}{(\beta_{mn})^{-\alpha}} (\gamma_3 \psi^{2\alpha+2} + (m-n)^2 \gamma_4 \psi^{2\alpha+4}) \right) (1+O(1)) \text{ for } \alpha \neq 0, \end{cases}$$

and

$$(2.10)' \quad \begin{cases} \psi^{1/2} \left(-J_0(\beta_{mn}\psi) \left[\gamma_0 \left(\frac{\psi^2}{2} \text{Log}(\beta_{mn}\psi) - \frac{\psi^2}{4} \right) + (m-n)^2 \right. \right. \\ \quad \times \left(\frac{\psi^4}{4} \text{Log}(\beta_{mn}\psi) - \frac{\psi^4}{16} \right) \left. \left. \right] + \frac{\pi}{2} Y_0(\beta_{mn}\psi) \right. \\ \quad \times \left. \left[\gamma_0 \frac{\psi^2}{2} + (m-n)^2 \frac{\psi^4}{4} \right] \right) (1+O(1)) \text{ for } \alpha = 0, \end{cases}$$

where $O(1)$ denotes here a function of ψ (and (m, n)) bounded by a constant for $(m, n; \psi) \in \mathcal{D}$ and γ_i ($i = 1, \dots, 4$) are absolute constants.

Now to determine the constants A_1 and A_2 corresponding to $v_{m,n}$, we fix (m, n) and divide both sides of (2.6) by $\psi^{\alpha+1/2}$. Let then $\psi \rightarrow 0$. We immediately obtain $A_2 = 0$ and $A_1 = 2^\alpha \Gamma(\alpha+1)/\beta_{mn}^\alpha$. Denoting by $r(m, n; \psi)$ the expression (2.10) for $\alpha \neq 0$ (respectively (2.10)' for $\alpha = 0$) divided by $\psi^{\alpha+1/2}$, we get

$$(2.11) \quad \left(\frac{\sin \psi}{\psi} \right)^{\alpha+1/2} (\cos \psi)^{1/2} R_{m,n}^\alpha(\cos \psi) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(\beta_{mn}\psi)}{(\beta_{mn}\psi)^\alpha} + r(m, n; \psi)$$

and Theorem 1 follows easily from (2.10) because $J_\nu(\beta_{mn}\psi)/(\beta_{mn}\psi)^\nu$ ($\nu = \pm\alpha$) is bounded for $(m, n; \psi) \in \mathcal{D}$ in the case $\alpha \neq 0$ and by (2.9)'-(2.10)' for $\alpha = 0$.

3. PROOF OF THE COROLLARY

3.1. LEMMA. Let $C > 0$. Then there is a number $C_3 > 0$ such that for every $(m, n; \psi)$ with $0 < 2\sqrt{mn}\psi \leq C$, we have

$$\left| \frac{J_\alpha(\beta_{mn}\psi)}{(\beta_{mn}\psi)^\alpha} - \frac{J_\alpha(2\sqrt{mn}\psi)}{(2\sqrt{mn}\psi)^\alpha} \right| \leq C_3(m+n)\psi^2.$$

PROOF. The function $f(x) = x^{-\alpha} J_\alpha(x)$ has a derivative $f'(x) = xg(x)$ where g is bounded by a constant $K > 0$ on $[0, C]$. Thus

$$|f(\beta_{mn}\psi) - f(2\sqrt{mn}\psi)| \leq K\beta_{mn}(\beta_{mn} - 2\sqrt{mn})\psi^2$$

for every $(m, n; \psi)$ such that $0 < 2\sqrt{mn}\psi \leq C$. The result then follows recalling (1.9).

3.2. By (1.8) and (3.1), for $(m, n; \psi) \in \mathcal{D}$, we have

$$\begin{aligned} & \left| R_{m,n}^\alpha(\cos \psi) - 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(2\sqrt{mn}\psi)}{(2\sqrt{mn}\psi)^\alpha} \right| \\ & \leq |R_{m,n}^\alpha(\cos \psi)| \cdot \left| 1 - \left(\frac{\sin \psi}{\psi} \right)^{\alpha+1/2} (\cos \psi)^{1/2} \right| \end{aligned}$$

$$+ \begin{cases} (C_1 + C_3(m+n))\psi^2 + C_2(m-n)^2\psi^4 & \text{for } \alpha \neq 0 \\ [(C_1 + C_3(m+n))\psi^2 + C_2(m-n)^2\psi^4](|\operatorname{Log} \beta_{mn}\psi| + 1) & \text{for } \alpha = 0. \end{cases}$$

Now with $x \in [0, C]$ and $\psi = x/2\sqrt{mn}$, the corollary follows by recalling Lemma 2.1 and letting $m \rightarrow +\infty$ and $n \rightarrow +\infty$.

4. REMARKS

4.1. The normalization factor in Mehler–Heine formula being $1/2\sqrt{mn}$ one might think perhaps it is sufficient to assume that $mn \rightarrow +\infty$ for (1.11) to be valid. This is not true as we can see by considering

$$\begin{aligned} R_{m,1}^\alpha \left(\cos \frac{x}{2\sqrt{m}} \right) &= \left(\cos \frac{x}{2\sqrt{m}} \right)^{m-1} R_1^{(\alpha, m-1)} \left(\cos \frac{x}{\sqrt{m}} \right) \\ &= \frac{1}{2} \left(\cos \frac{x}{2\sqrt{m}} \right)^{m-1} \cdot \left(\frac{\alpha+m+1}{\alpha+1} \cos \frac{x}{\sqrt{m}} + \frac{\alpha-m+1}{\alpha+1} \right). \end{aligned}$$

When $m \rightarrow +\infty$, the limit of this expression is not given by (1.11).

4.2. When $\alpha \geq 0$ and $\beta \geq 0$, our result contains the classical formula (1.1) as a particular case. Indeed, by (1.6) we have

$$R_{n,n+\beta}^\alpha \left(\cos \frac{x}{2\sqrt{n(n+\beta)}} \right) = \left(\cos \frac{x}{2\sqrt{n(n+\beta)}} \right)^\beta R_n^{(\alpha, \beta)} \left(\cos \frac{x}{\sqrt{n(n+\beta)}} \right).$$

4.3. If $\alpha = d-2 > 0$ ($d \in \mathbb{N}$), the polynomials $R_{m,n}^\alpha$ on D can be interpreted as the zonal functions of the Gelfand pair $(U(d), U(d-1))$ ([2]). In this case our formula can be compared to the case of a symmetric space studied in [3].

5. PROOF OF THEOREM 2

Let $A = [a, b] \times [c, d]$ ($a > 0, c > 0$) and suppose first that $(m_0, n_0) = (0, 0)$. Let N be fixed. By (1.16) we have

$$(5.1) \quad \begin{cases} \mu_N(A) = P^N((0, 0), \sqrt{N}A) \\ = \sum_{(m,n) \in \sqrt{N}A} \Pi_{m,n}^\alpha \iint_D x^N \overline{R_{m,n}^\alpha(z)} \frac{\alpha+1}{\pi} (1-|z|^2)^\alpha dx dy. \end{cases}$$

Then let $z = \cos \psi e^{i\theta}$ ($\psi \in [0, \pi/2]$, $\theta \in [0, 2\pi]$) and note that by using (1.6) we can restrict to $[0, \pi/2]$ the integration in the θ variable if in the summation we only consider the values (m, n) such that $N+m-n$ is even. Finally changing θ by θ/\sqrt{N} and ψ by ψ/\sqrt{N} we get

$$(5.2) \quad \left\{ \begin{aligned} \mu_N(A) &= \frac{4(\alpha+1)}{\pi} \frac{1}{N} \sum_{\mathcal{A}_N} \Pi_{m,n}^\alpha \int_0^{\pi\sqrt{N}/2} \left(\cos \frac{\theta}{\sqrt{N}} \right)^N e^{i(m-n)(\theta/\sqrt{N})} d\theta. \\ &\cdot \int_0^{\pi\sqrt{N}/2} \left(\cos \frac{\psi}{\sqrt{N}} \right)^{N+1} R_{m,n}^\alpha \left(\cos \frac{\psi}{\sqrt{N}} \right) \\ &\cdot \left(\frac{\sin \psi/\sqrt{N}}{\psi/\sqrt{N}} \right)^{2\alpha+1} \left(\frac{\psi}{\sqrt{N}} \right)^{2\alpha+1} d\psi, \end{aligned} \right.$$

where $\mathcal{A}_n = \{(m, n) \in \mathbb{N}^2 \mid (m, n) \in \sqrt{N}A \text{ and } N+m-n \text{ even}\}$. For every number $C > 0$, we write

$$\mu_N(A) = I_1(N, C) + I_2(N, C),$$

where $I_1(N, C)$ (resp. $I_2(N, C)$) is the same expression as in (5.2) but the integration in the ψ variable is over $[0, C]$ (resp. $[C, \pi\sqrt{N}/2]$).

5.3. LEMMA.

$$\lim_{C \rightarrow +\infty} (\sup_{N > 0} I_2(N, C)) = 0.$$

PROOF. A trivial calculation yields to

$$(5.4) \quad \Pi_{m,n}^\alpha = \frac{(m+n)m^\alpha n^\alpha}{\Gamma(\alpha+1)\Gamma(\alpha+2)} \left(1 + O\left(\frac{1}{m+n}\right) \right),$$

and for θ and ψ in $[0, \pi\sqrt{N}/2]$ we clearly have

$$(5.5) \quad \left(\cos \frac{\theta}{\sqrt{N}} \right)^N \leq e^{-\theta^2/2} \quad \text{and} \quad \left(\cos \frac{\psi}{\sqrt{N}} \right)^N \leq e^{-\psi^2/2}.$$

So we easily obtain

$$|I_2(N, C)| \leq K \frac{1}{N} \sum_{\mathcal{A}_N} \frac{(m+n)m^\alpha n^\alpha}{(\sqrt{N})^{2\alpha+1}} \int_0^{\pi\sqrt{N}/2} e^{-\theta^2/2} d\theta \int_C^{\pi\sqrt{N}/2} e^{-\psi^2/2} \psi^{2\alpha+1} d\psi,$$

where K is a constant. But

$$\frac{1}{N} \sum_{\mathcal{A}_N} \left(\frac{m+n}{\sqrt{N}} \right) \left(\frac{m}{\sqrt{N}} \frac{n}{\sqrt{N}} \right)^\alpha$$

is bounded independently of N because it is a Riemann sum over A of the function $(x, y) \rightarrow (x+y)(xy)^\alpha$. The result of the lemma follows immediately.

5.6. LEMMA.

$$\lim_{C \rightarrow +\infty} (\lim_{N \rightarrow +\infty} I_1(N, C)) = \frac{2^{\alpha+1}}{\sqrt{2\pi}\Gamma(\alpha+1)} \iint_A (x+y)(xy)^\alpha e^{-(1/2)(x+y)^2} dx dy.$$

PROOF. For $(m, n) \in \sqrt{N}A$ and $\psi \in [0, C]$, we have $\psi/\sqrt{N} \leq C\sqrt{bd}/\sqrt{mn}$. So we can replace in $I_1(N, C)$, $R_{m,n}^\alpha(\cos \psi/\sqrt{N})$ by its expression in terms of Theorem 1. By Lemma 3.1 and formula (5.4) this yields to

$$(5.7) \quad \left\{ \begin{aligned} I_1(N, C) &= \frac{4}{\pi \Gamma(\alpha+1)} \int_0^{\pi \sqrt{N}/2} \int_0^C \sum_{\mathcal{A}_N} \frac{1}{N} \left(\frac{m+n}{\sqrt{N}} \right) \left(\frac{m}{\sqrt{N}} \frac{n}{\sqrt{N}} \right)^{\alpha/2} \\ &\cdot \left(1 + O\left(\frac{1}{m+n} \right) \right) \left(\cos \frac{\theta}{\sqrt{N}} \right)^N \cdot e^{i(m-n)/\sqrt{N}\theta} \\ &\cdot \left[J_\alpha \left(2 \left(\frac{m}{\sqrt{N}} \frac{n}{\sqrt{N}} \right)^{1/2} \psi \right) + \frac{(mn)^{\alpha/2}}{\Gamma(\alpha+1)} \left(\frac{\psi}{\sqrt{N}} \right)^\alpha \varepsilon(m, n; \psi/\sqrt{N}) \right] \\ &\cdot \psi^{\alpha+1} \left(\cos \frac{\psi}{\sqrt{N}} \right)^{N+1/2} \cdot \left(\frac{\sin \psi/\sqrt{N}}{\psi/\sqrt{N}} \right)^{\alpha+1/2} d\theta d\psi, \end{aligned} \right.$$

where

$$(5.8) \quad |\varepsilon(m, n; \psi/\sqrt{N})| \leq C_1'(m+n)\psi^2/N + C_2(m-n)^2\psi^4/N^2 \quad \text{for } \alpha \neq 0,$$

$$(5.8)' \quad \left\{ \begin{aligned} |\varepsilon(m, n; \psi/\sqrt{N})| &\leq [C_1'(m+n)\psi^2/N + C_2(m-n)^2\psi^4/N^2] \\ &\cdot \left(\left| \text{Log } 2\sqrt{mn} \frac{\psi}{\sqrt{N}} \right| + 1 \right) \quad \text{for } \alpha = 0. \end{aligned} \right.$$

It is then easily seen that the principal part of $I_1(N, C)$ (i.e. (5.7) without the ε -term) converges as $N \rightarrow +\infty$ to

$$(5.9) \quad \left\{ \begin{aligned} &\frac{4}{\pi \Gamma(\alpha+1)} \int_0^{+\infty} \int_0^C \left(\frac{1}{2} \iint_A (x+y)(xy)^{\alpha/2} e^{i(x-y)\theta} J_\alpha(2\sqrt{xy}\psi) dx dy \right) \\ &\cdot e^{-\theta^2/2} \psi^{\alpha+1} e^{-\psi^2/2} d\theta d\psi, \end{aligned} \right.$$

the dominated convergence being justified by inequalities (5.5). By interchanging summations in (5.9) we obtain the value

$$(5.10) \quad \left\{ \begin{aligned} &\frac{2}{\pi \Gamma(\alpha+1)} \iint_A (x+y)(xy)^{\alpha/2} \sqrt{\frac{\pi}{2}} e^{-(x-y)^2/2} \\ &\cdot \int_0^C \psi^{\alpha+1} J_\alpha(2\sqrt{xy}\psi) e^{-\psi^2/2} d\psi dx dy. \end{aligned} \right.$$

But if $C = +\infty$, the integral in ψ is equal to $2^\alpha(xy)^{\alpha/2} e^{-2xy}$ (see [8] p. 93) and this gives the result of the lemma for the principal part of $I_1(N, C)$. At last let's show that the remainder term tends to zero as $N \rightarrow +\infty$. It is equal to

$$(5.11) \quad \left\{ \begin{aligned} &Cte \int_0^{\pi \sqrt{N}/2} \int_0^C \left[\sum_{\mathcal{A}_N} e^{i(m-n)(\theta/\sqrt{N})} \varepsilon(m, n; \psi/\sqrt{N}) \frac{(m+n)(nm)^\alpha}{N^{\alpha+3/2}} \right. \\ &\quad \cdot \left(1 + O\left(\frac{1}{m+n} \right) \right) \Big] \\ &\quad \cdot \left(\cos \frac{\theta}{\sqrt{N}} \right)^N \cdot \left(\cos \frac{\psi}{\sqrt{N}} \right)^{N+1/2} \cdot \left(\frac{\sin \psi/\sqrt{N}}{\psi/\sqrt{N}} \right)^{\alpha+1/2} \cdot \psi^{2\alpha+1} d\theta d\psi. \end{aligned} \right.$$

But by (5.8) the sum into the brackets is bounded by

$$(5.12) \quad \frac{2}{\sqrt{N}} \sum_{(m,n) \in \sqrt{N}A} \frac{1}{N} \left(C_1'(m+n) \frac{\psi^2}{N} + C_2(m-n)^2 \frac{\psi^4}{N^2} \right) \frac{(m+n)(mn)^\alpha}{N^\alpha} \quad (\alpha \neq 0)$$

and this tends to zero as $N \rightarrow +\infty$ because the sum in (5.12) consists in a vanishing term plus a Riemann sum over A of the function $(x, y) \rightarrow C_1'(x+y)^2 \psi^2(xy)^\alpha$. Recalling (5.5), the dominated convergence theorem implies that (5.11) tends to zero as $N \rightarrow +\infty$ and Lemma 5.6 is proved for $\alpha \neq 0$. If $\alpha = 0$, we use (5.8)' and we obtain the result in the same way.

5.13. END OF THE PROOF OF THEOREM 2. Denote by l the value of the limit in Lemma 5.6. For a fixed $C > 0$, we have clearly

$$|\mu_N(A) - l| \leq |I_2(N, C)| + |I_1(N, C) - \lim_{N \rightarrow +\infty} I_1(N, C)| + |\lim_{N \rightarrow +\infty} I_1(N, C) - l|.$$

Let $\varepsilon > 0$ be given. By Lemmata 5.3 and 5.6, we can choose C such that the first and the third terms in the above inequality are less than $\varepsilon/3$ (independently of N).

Then for N large enough, the second term is also less than $\varepsilon/3$ and the theorem is proved in the case $(m_0, n_0) = (0, 0)$. If $(m_0, n_0) \neq (0, 0)$, there is the additional term $R_{m_0, n_0}^\alpha(z)$ in the integral of formula (5.1) but the computation follows exactly the same lines.

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